

# Periodic Orbit Theory of Anomalous Diffusion

Roberto Artuso

*Dipartimento di Fisica dell'Università and I.N.F.N.  
Via Celoria 16, I-20133 Milano, Italy*

Giulio Casati

*Dipartimento di Fisica dell'Università and I.N.F.N.  
Via Castelnovo, I-22100 Como, Italy*

and

Roberto Lombardi

*Dipartimento di Fisica dell'Università  
Via Celoria 16, I-20133 Milano, Italy*

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## Abstract

We introduce a novel technique to find the asymptotic time behaviour of deterministic systems exhibiting anomalous diffusion. The procedure is tested for various classes of simple but physically relevant 1-D maps and possible relevance of our findings for more complicated problems is briefly discussed.

In the last few years the phenomenon of deterministic diffusion has been widely investigated: as a matter of fact a major theoretical challenge in the study of dynamical systems is to understand thoroughly the generation of typical stochastic properties in purely deterministic systems. Moreover even the simplest maps for which deterministic diffusion has been observed are supposed to model the behaviour of relevant physical systems (like Josephson junctions in presence of a microwave field, see e.g. [1]). A recent approach, introduced independently in [2] and [3], leads to an expression for the diffusion coefficient in terms of the periodic orbits of the system, in the form of a cycle expansion<sup>[4]</sup>. Cycle expansions have been applied in a number of different contexts (see for instance [5, 6]): they work remarkably well for low dimensional hyperbolic systems, provided their topology (symbolic dynamics) is under control. The problem of controlling the topology of the system is well illustrated, in the context of diffusion properties via cycle expansions, when one deals with the infinite Lorentz gas with bounded horizon (see [7]). If one relaxes the hypothesis of pure hyperbolicity (absence of marginal stability), much more care has to be taken, and generally the effectiveness of the expansions is regained only if one is able to sum infinite contributions shadowing the marginal fixed point<sup>[5]</sup>. We emphasize that this problem is of paramount relevance, as it naturally arises when dealing with generic hamiltonian systems, in which elliptic islands of stability and hyperbolic homoclinic webs coexist.

Here we address explicitly the problem of marginal stability, which is tightly connected to the appearance of anomalous diffusion (and also represents a crucial feature in establishing a good theory for diffusion in generic two-dimensional area preserving maps<sup>[8]</sup>). We extend the theory developed in [2, 3] to study deviations from normal diffusion, expressing the asymptotic time behaviour in terms of properties of the periodic orbits of the system. The method is then applied to a class of 1-D maps for which anomalous diffusion has been previously observed.

We begin by recalling the essential features of the periodic orbit theory of normal diffusion<sup>[2, 3]</sup>, by considering the simplest context in which it may be applied. We will consider lifts of one-dimensional circle maps

$$x_{t+1} = f(x_t) \quad t \in \mathbf{N} \quad f(x+n) = n + f(x) \quad f(-x) = -f(x) \quad (1)$$

together with corresponding torus maps  $x_{t+1} = f(x_t)|_{mod 1} = \hat{f}(x_t)$ . Normal diffusion means that asymptotically  $\sigma^2(t) = \langle (x_t - x_0)^2 \rangle \sim 2Dt$ , where the average is over initial conditions (by symmetry  $x_0$  may be taken in the unit interval). The set of

all periodic orbits of  $\hat{f}$  will be denoted by  $\{p\}$ : each orbit will be characterized by its period  $n_p$  stability  $\Lambda_p$  (product of the derivatives along the cycle) and integer winding number  $\sigma_p$  (such that for each cycle point  $x_{i(p)}$  we have  $f^{n_p}(x_{i(p)}) = x_{i(p)} + \sigma_p$ ). We focus our attention on the generating function  $\langle e^{\beta(x_t - x_0)} \rangle = \Omega_t(\beta)$ ; it has been shown<sup>[2, 3]</sup> that its asymptotic behaviour is dominated by the leading eigenvalue of an appropriate transfer operator:  $\Omega_t(\beta) \sim z(\beta)^{-t}$ , where  $z(\beta)$  is the smallest solution of

$$\zeta_0^{-1}(z(\beta), \beta) = \prod_{\{p\}} \left( 1 - \frac{z^{n_p} \exp(\sigma_p \beta)}{|\Lambda_p|} \right) = 0$$

Expansion of the generating function for small  $\beta$  thus yields

$$D = -\frac{1}{2} \frac{d^2}{d\beta^2} z(\beta) \Big|_{\beta=0}$$

Anomalous diffusion is associated to a vanishing or diverging  $D$ . In order to generalize the theory we take into account that, up to time  $t$ ,  $\Omega_t(\beta)$  gets contributions from periodic orbits with  $n_p \leq t$ <sup>[2, 3]</sup> (as it is connected to the trace of the  $t$ -th power of the transfer operator): we thus may write in general

$$\Omega_t(\beta) \sim [z^{[t]}(\beta)]^{-t} \quad (2)$$

where  $z^{[t]}(\beta)$  is such that  $\zeta_{0[t]}^{-1}(z^{[t]}(\beta), \beta) = 0$ ,  $\zeta_{0[t]}^{-1}$  denoting the cycle expansion of  $\zeta_0^{-1}$  truncated to finite order  $t$ . From (2) we thus get the asymptotic behaviour

$$\sigma^2(t) \sim -t \frac{\partial^2 / \partial \beta^2 \zeta_{0[t]}^{-1}(z, \beta)}{\partial / \partial z \zeta_{0[t]}^{-1}(z, \beta)} \Big|_{z=1, \beta=0} \quad (3)$$

where we took into account probability conservation<sup>[4]</sup> ( $z^{[t]}(0) \rightarrow 1$  as  $t \rightarrow \infty$ ) as well as symmetry of the map. This procedure may be alternatively interpreted as follows: for a sequence of large increasing times  $t_k$  we approximate the system by a sequence of hyperbolic (regularly diffusing) systems  $S_k$  with polynomial zeta functions  $\zeta_0^{-1}(z, \beta)_{S_k} = \zeta_{0[t_k]}^{-1}(z, \beta)$ :  $\sigma^2(t) \sim 2D(t_k) \cdot t$  for  $t \lesssim t_k$  (for both the original system and  $S_k$ ), thus corrections to normal diffusion are connected with the asymptotic behaviour of  $D(t)$ .

To test the theory we consider a map of the form (1) (see fig. 1) characterized by the presence of a marginal fixed point<sup>[9, 10]</sup>. The corresponding map on the torus  $\hat{f}$  is shown in fig. 2, and consists of five complete branches. The domain

is accordingly partitioned into five subsets, which we will denote respectively by 1, 2, 0, 3, 4. Branches 1, 2, 3, 4 have a constant absolute value of the slope  $\Lambda$ , while in region 0 the map takes the usual Manneville and Pomeau form<sup>[11]</sup>  $x_{n+1} = x_n + a \cdot x_n^\gamma$  ( $\gamma > 1$ ), leading to intermittency. This partition induces a good symbolic dynamics: every possible combination of symbols is physically realized as a trajectory of the map.

To apply the formalism we have to accomplish two things: enumerate all possible cycles once the marginal fixed point is pruned away (the effect of the marginal fixed point is nonlinear and is probed by the infinity of cycles which accumulate to it<sup>[5]</sup>), and specify  $\sigma_p$  and  $|\Lambda_p|$  for each cycle. It is easy to see that the symbolic dynamics corresponding to prohibiting an infinite repetition of 0 is determined by unrestricted grammar in the following (countable) alphabet  $\{0^k 1, 0^l 2, 0^n 3, 0^i 4, 1, 2, 3, 4\}$  ( $k, l, n, i = 1, 2, 3, \dots$ ). As regards winding numbers, if we denote by  $\epsilon_j$  a generic letter of the alphabet we have  $\sigma_{\epsilon_1 \epsilon_2 \dots \epsilon_n} = \sum_{i=1}^n \sigma_{\epsilon_i}$  where for each letter including symbols 1 or 2  $\sigma_\epsilon = 1$ , while for each letter including symbols 3 or 4  $\sigma_\epsilon = -1$ . We now use the piecewise linear approximation of Gaspard and Wang<sup>[12]</sup> to the Manneville Pomeau system to estimate the stability of cycles shadowing the marginal fixed point, so that  $|\Lambda_{\epsilon_1 \epsilon_2 \dots \epsilon_n}| = \prod_{j=1}^n |\Lambda_{\epsilon_j}|$ , and for the various letters we have

$$\begin{aligned} |\Lambda_\epsilon| &= \Lambda & \epsilon &= 1, 2, 3, 4 \\ |\Lambda_\epsilon| &= \Lambda \cdot k^{\alpha+1} & \epsilon &= 0^k L \quad L = 1, 2, 3, 4 \end{aligned} \quad (4)$$

with  $\alpha = 1/(\gamma - 1)$ . As each curvature<sup>[13]</sup> is zero we get that the zeta function is expressed in terms of the fundamental cycles<sup>[4]</sup> (which are determined by the letters) and takes the following form

$$\zeta_0^{-1}(z, \beta) = 1 - \frac{4 \cosh \beta}{\Lambda} z \left( 1 + \sum_{k=1}^{\infty} \frac{z^k}{k^{\alpha+1}} \right)$$

In this way (3) reads

$$\sigma^2(t) \sim t \frac{\sum_{k=1}^t k^{-(\alpha+1)}}{\sum_{k=1}^t k^{-\alpha}}$$

This leads to normal diffusivity for  $\alpha > 1$ , while for other parameter values we get

$$\sigma^2(t) \sim \begin{cases} t^\alpha & \text{for } \alpha \in (0, 1) \\ t / \ln t & \text{for } \alpha = 1 \end{cases} \quad (5)$$

in agreement with probabilistic estimates and numerical simulations in [9]. We remark that our findings are entirely based on metric and topological properties of

the system, thus the relation, expressed by (5), between anomalous diffusion and exponents of power-law stabilities does not rely on any stochastic modelization of the system.

Our theory may also be applied when the diffusion is accelerated<sup>[10]</sup>, that is when marginal fixed points appear in the running branches (see fig. 3). We will give a detailed treatment of the results elsewhere: here we just quote the results (in agreement with [10], except the case  $\alpha = 1$ )

$$\sigma^2(t) \sim \begin{cases} t & \text{for } \alpha > 2 \\ t \ln t & \text{for } \alpha = 2 \\ t^{3-\alpha} & \text{for } \alpha \in (1, 2) \\ t^2 / \ln t & \text{for } \alpha = 1 \\ t^2 & \text{for } \alpha \in (0, 1) \end{cases} \quad (6)$$

where  $\alpha$  is still  $1/(\gamma - 1)$ . We observe that topological intricacies of the system in its laminar phase do not affect the asymptotic estimates (5) and (6), which are exclusively determined by the sequence of cycles accumulating to the marginal fixed points. It is precisely this robustness with respect to fine details of the dynamics which let us claim that our approach may be relevant also in much more complex systems, like the Lorentz gas with infinite horizon. This may be appreciated by a simple argument: the weight of orbits travelling a time  $t$  without collisions in this system goes like  $t^{-3}$ <sup>[14]</sup>, corresponding to  $\alpha = 2$  in (4). From (6) we thus get  $\sigma^2(t) \sim t \ln t$ , which indeed seems to reproduce the correct behaviour<sup>[15]</sup>.

In this paper we have proposed an extension of the periodic orbit theory of deterministic diffusion<sup>[2, 3]</sup>, capable of predicting the asymptotic time behaviour of  $\sigma^2(t)$  by using an appropriate cycle expansion. In particular this technique has been tested with classes of one-dimensional maps in which intermittency slows or accelerates diffusion: the anomalous diffusion is completely characterized by the exponent  $\alpha$  ruling power-law stability of cycles shadowing the marginal fixed points ( $\alpha$  is in turn connected to the intermittency exponent  $\gamma$ ). Our results provide examples in which cycle expansions can successfully deal with marginal stability: this supports the view that this technique may represent a major tool in the analysis of generic low-dimensional (both classical and quantum) hamiltonian systems.

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## References

- [1] B.A. Huberman, J.P. Crutchfield and N.H. Packard, *Appl.Phys.Lett.* **37** (1980) 750; M. Cirillo and N.F. Pedersen, *Phys.Lett.* **90A** (1982) 150; E. Ben-Jacob, J. Goldhirsch, Y. Imry and S. Fishman, *Phys.Rev.Lett.* **49** (1982) 1599
- [2] R. Artuso, *Phys.Lett.* **160A** (1991) 528
- [3] P. Cvitanović, J.-P. Eckmann and P. Gaspard, Transport Properties of the Lorentz Gas in Terms of Periodic Orbits, *Niels Bohr Institute preprint* (May 1991)
- [4] R. Artuso, E. Aurell and P. Cvitanović, *Nonlinearity* **3** (1990) 325
- [5] R. Artuso, E. Aurell and P. Cvitanović, *Nonlinearity* **3** (1990) 361
- [6] P. Cvitanović and B. Eckhardt, *Phys.Rev.Lett.* **63** (1989) 823; R. Mainieri, *Phys.Rev.* **A45** (1992) 3580; G. Tanner, P. Scherer, E.B. Bogomolny, B. Eckhardt and D. Wintgen, *Phys.Rev.Lett.* **67** (1991) 2410; F. Christiansen, G. Paladin and H.H. Rugh, *Phys.Rev.Lett.* **65** (1990) 361
- [7] P. Cvitanović, P. Gaspard and T. Schreiber, *Chaos* **2** (1992) 85
- [8] B. Eckhardt, Periodic Orbits and Diffusion in Standard Maps, *Marburg preprint* (July 1992)
- [9] T. Geisel and S. Thomae, *Phys.Rev.Lett.* **52** (1984) 1936
- [10] T. Geisel, J. Nierwetberg and A. Zacherl, *Phys.Rev.Lett.* **54** (1985) 616
- [11] Y. Pomeau and P. Manneville, *Commun.Math.Phys.* **74** (1980) 189; P. Manneville, *J.Phys. (Paris)* **41** (1980) 1235
- [12] P. Gaspard and X.-J. Wang, *Proc.Natl.Acad.Sci. U.S.A.* **85** (1988) 4591; X.-J. Wang, *Phys.Rev.* **A40** (1989) 6647
- [13] P. Cvitanović, *Phys.Rev.Lett.* **61** (1988) 2729
- [14] L.A. Bunimovich, *Sov.Phys. JETP* **62** (1985) 842
- [15] B. Friedman and R.F. Martin, Jr., *Phys.Lett.* **105A** (1984) 23; A. Zacherl, T. Geisel, J. Nierwetberg and G. Radons, *Phys.Lett.* **114A** (1986) 317; P.M. Bleher, *J.Stat.Phys.* **66** (1992) 315

### Figure captions

1. Example of a circle map with a marginal fixed point.
2. Map on the torus associated to the one shown in fig. 1.
3.  $\hat{f}$  for a model of accelerated diffusion: marginal fixed points are situated in the running branches ( $|\sigma| = 1$ ).